Biostatistics I: Hypothesis testing

Continuous data: Two-sample (dependent) tests

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In this Section

- Two-sample t-test (dependent samples)
- Two-sample Wilcoxon signed-rank test (dependent samples)
- Examples

Two-sample t-test (dependent samples): Theory

Assumptions

- The dependent variables are continuous
- The observations are dependent
- The distribution of the differences in the variable between the two correlated groups are approximately normally distributed
- The dependent variables do not contain any outliers

Two-sample t-test (dependent samples): Theory

Scenario

Is the mean BMI of the students before the exams different from the mean BMI of the students after the exams?

Connection with linear regression

$y_{2i} - y_{1i} = \beta_0 + \beta_1 x_i + \epsilon_i$, where $x_i = 0$

$H_0 : \beta_0 = 0$
$H_1 : \beta_0 \neq 0$

Two-sample t-test (dependent samples): Theory

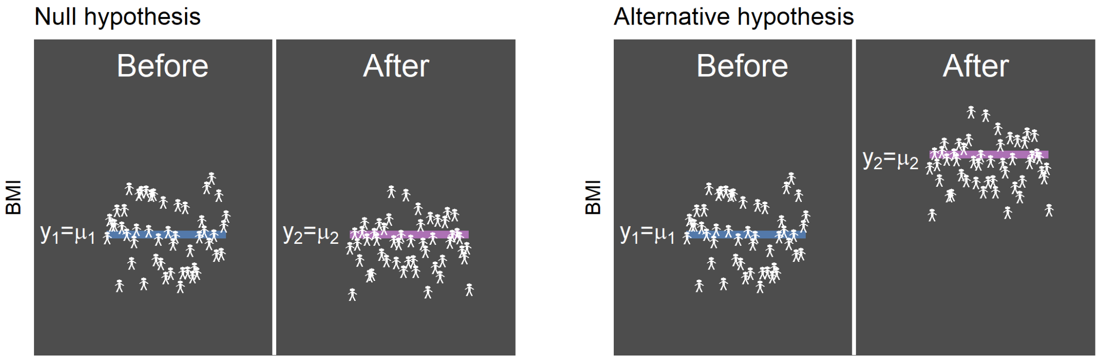
Scenario

Is the mean BMI of the students in my university different from the BMI of all students?

Alternatively

$$H_0 : \mu_1 - \mu_2 = 0$$
$$H_1 : \mu_1 - \mu_2 \neq 0$$

Two-sample t-test (dependent samples): Theory



Two-sample t-test (dependent samples): Theory

Test statistic

$t = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{sd(x)/\sqrt{n}}$, where

- Sample mean of group 1: $\bar{x}_1$
- Sample mean of group 2: $\bar{x}_2$
- Standard deviation of the difference: $sd(x)$
- Number of subjects: n

Sampling distribution

- t-distribution with $df = n - 1$
- Critical values and p-value

Type I error

- Normally $\alpha = 0.05$

Draw conclusions

- Compare test statistic (t) with the critical values or the p-value with α

Two-sample t-test (dependent samples): Application

Scenario

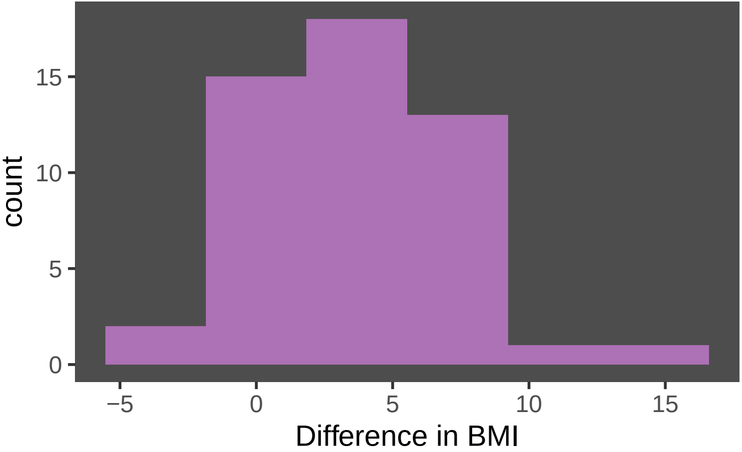
Is the mean BMI of the students before the exams different from the mean BMI of the students after the exams?

Hypothesis

$H_0 : \mu_1 - \mu_2 = 0$
$H_1 : \mu_1 - \mu_2 \neq 0$

Two-sample t-test (dependent samples): Application

Collect and visualize data



Two-sample t-test (dependent samples): Application

Hypothesis

$H_0 : \mu_1 - \mu_2 = 0$
$H_1 : \mu_1 - \mu_2 \neq 0$

Test statistic

Let's assume that:

- Sample mean before: $\bar{x}_1 = 24$
- Sample mean after: $\bar{x}_2 = 25$
- Standard deviation of the difference: $sd(x) = 4$
- Number of subjects: $n = 50$

$t = \frac{\bar{x}_1 - \bar{x}_2}{sd(x)\sqrt{n}} = \frac{24-25}{4/\sqrt{(50)}} = -1.77$

Degrees of freedom

$df = n - 1 = 49$

Type I error

$\alpha = 0.05$

Two-sample t-test (dependent samples): Application

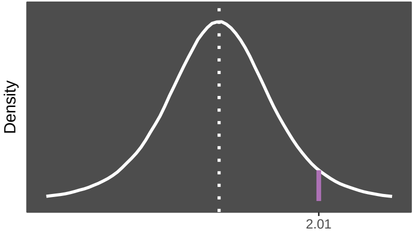
Critical values

Using R we get the critical values from the *t*-distribution:

critical value$_{\alpha/2}$ = critical value$_{0.05/2}$

```
qt(p = 0.05/2, df = 49, lower.tail = FALSE)
```

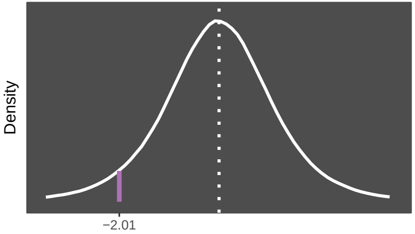
```
[1] 2.009575
```



-critical value$_{\alpha/2}$ = -critical value$_{0.05/2}$

```
qt(p = 0.05/2, df = 49, lower.tail = TRUE)
```

```
[1] -2.009575
```



Two-sample t-test (dependent samples): Application

Draw conclusions

We reject the H_0 if:

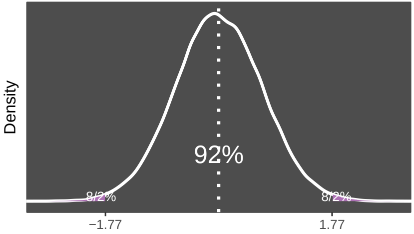
- t > critical value$_{\alpha/2}$ or t < - critical value$_{\alpha/2}$

We have -1.77 > -2.01 $\Rightarrow$ we do not reject the H_0

Using R we obtain the p-value from the t-distribution:

```
2 * pt(q = -1.77, df = 49, lower.tail = TRUE)
```

```
[1] 0.08294898
```



Two-sample Wilcoxon signed rank test: Theory

Assumptions

- Population distribution of the differences is symmetric
- The observations are dependent

Two-sample Wilcoxon signed rank test: Theory

Scenario

Is the median score value of the students in my university different this year compared to next year?

Connection with linear regression

$signed\_rank(y_{2i} - y_{1i}) = \beta_0 + \beta_1 x_i + \epsilon_i$, where $x_i = 0$

$H_0 : \beta_0 = 0$
$H_1 : \beta_0 \neq 0$

Alternatively

$H_0 : m_1 = m_2$
$H_1 : m_1 \neq m_2$

Two-sample Wilcoxon signed rank test: Theory

Test statistic

- Calculate the ranks of the absolute difference
  - If there are ties you assign the average of the tied ranks
  - If a pair of scores is equal then they are considered tied and dropped from the analysis and the sample size is reduced
- Obtain the sum of those ranks where the difference was positive $W_+ = \sum R_d^+$ or negative $W_- = \sum R_d^-$. The test statistic (W) is the minimum of W_+ and W_-

If **one-tailed**: use either W_+ or W_- for the test statistic (W) depending on the direction of the alternative hypothesis

Two-sample Wilcoxon signed rank test: Theory

Sampling distribution

For large sample size: we can use the normal approximation, that is, W is normally distributed.

$$\mu_W = \frac{n(n+1)}{4}$$
$$\sigma_W = \sqrt{\frac{n(n+1)(2n+1)}{24}}$$
$$z = \frac{|max(W_+, W_-) - \mu_W| - 1/2}{\sigma_W}$$

When there are ties, the mean stays the same but the variance is reduced by a quantity

For small sample size: we can use the exact distribution (more details in the application).

- Get critical values and p-value

Two-sample Wilcoxon signed rank test: Theory

Type I error

- Normally $\alpha = 0.05$

Draw conclusions

- Compare test statistic with the critical values or the p-value with α

Two-sample Wilcoxon signed rank test: Application

Scenario

Is the median score value of the students in my university different this year compared to next year?

Hypothesis

$H_0 : m_1 = m_2$
$H_1 : m_1 \neq m_2$

Two-sample Wilcoxon signed rank test: Application

Collect and visualize data

x	y	Difference	|Difference|	rank
5	10	-5	5	3
4	5	-1	1	1
4	8	-4	4	2

Hypothesis

$H_0 : m_1 = m_2$

$H_1 : m_1 \neq m_2$

Test statistic

$W_- = 6$ and $W_+ = 0$

Type I error

$\alpha = 0.05$

Two-sample Wilcoxon signed rank test: Application

Critical values

Using R we get the critical values from the exact distribution:
low critical value$_{\alpha/2}$ = low critical value$_{0.05/2}$

```
qsignrank(p = 0.05/2, n = 3, lower.tail = TRUE)
```

```
[1] 0
```

high critical value$_{\alpha/2}$ = high critical value$_{0.05/2}$

```
qsignrank(p = 0.05/2, n = 3, lower.tail = FALSE)
```

```
[1] 6
```

Two-sample Wilcoxon signed rank test: Application

Draw conclusions

We reject the H_0 if:

- $W >$ high critical value$_{\alpha/2}$ or $W <$ low critical value$_{\alpha/2}$

We have 6 = 6 and 0 = 0 $\Rightarrow$ we do *not* reject the H_0

Two-sample Wilcoxon signed rank test: Application

Draw conclusions

Using R we obtain the p-value from the exact distribution:

$p - value = 2 * Pr(W <= 0):$

```
2 * psignrank(q = 0, n = 3)
```

```
[1] 0.25
```

or

$p - value = 2 * Pr(W >= 6) = 2 * (1 - Pr(W < 6)):$

```
2 * (1 - psignrank(q = 6 - 1, n = 3, lower.tail = TRUE))
```

```
[1] 0.25
```

```
2 * psignrank(q = 6 - 1, n = 3, lower.tail = FALSE)
```

```
[1] 0.25
```